Ch 11. Stabilization and State Estimation via Sliding Mode Control

Goal : Sliding mode control, Sliding mode observer for

linear time-invariant singularly perturbed systems subject to impulsive effects.

$$(*)\begin{cases} \dot{x} = A_{11}x + A_{12}z + Bu, & t \neq \tau_k \\ \varepsilon\dot{z} = A_{21}x + A_{22}z, & t \neq \tau_k \\ x(t) = (I + E_k)\,x(t^-), & t = \tau_k \\ z(t) = (I + F_k)\,z(t^-), & t = \tau_k \\ x(0) = x_0,\ z(0) = z_0 \end{cases}$$

($x \in \mathbb{R}^n$: slow state, $z \in \mathbb{R}^m$: fast state, $u \in \mathbb{R}^r$: feedback control,

$A_{11} \in \mathbb{R}^{n\times n}$, $A_{12} \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{n\times r}$, $A_{21} \in \mathbb{R}^{m\times n}$, $A_{22} \in \mathbb{R}^{m\times m}$, $E_k \in \mathbb{R}^{n\times n}$, $F_k \in \mathbb{R}^{m\times m}$,

$\tau_k \nearrow \infty$.)

$(*)$: globally exponentially stabilized by u if $\|x(t)\| + \|z(t)\| \leq k(\|x_0\| + \|z_0\|)e^{-\lambda t}$

① Consider slow reduced, nonimpulsive subsystem.

(→ later stabilize the whole system)

For $t \neq \tau_k$, setting $\varepsilon = 0$ yields

$$\begin{cases} \dot{x} = A_{11}x + A_{12}z + Bu \\ 0 = A_{21}x + A_{22}z \end{cases}$$

$\Rightarrow$ $\underline{z = h(x) = -A_{22}^{-1} A_{21} x}$ (We assume A_{22} : nonsingular)

$\boxed{\text{Reduced subsystem}}$ $\dot{x}_s = A_0 x_s + B_0 u_s$, $\underbrace{A_0 = A_{11} - A_{12} A_{22}^{-1} A_{21}}_{\in \mathbb{R}^{n\times n}}$, $\underbrace{B_0 = B}_{\in \mathbb{R}^{n\times m}}$

- We consider r-dim sliding mode hyper-surface defined by

$$S_s(x_s) = \begin{pmatrix} s_1(x_s) \\ \vdots \\ s_r(x_s) \end{pmatrix} = \begin{pmatrix} c_1^T x_s \\ \vdots \\ c_r^T x_s \end{pmatrix} = \begin{pmatrix} c_1^T \\ \vdots \\ c_r^T \end{pmatrix} x_s = \underbrace{C_s}_{\mathbb{R}^{r\times n}} x_s$$

$$\Rightarrow \left(\frac{d}{dt} S_s(x_s) = \right) \dot{S}_s(x_s(t)) = C_s \dot{x}_s(t) = C_s A_0 x_s + C_s B_0 u_s$$

r-dim equivalent control : $\underline{u_s^{eq} = -(C_s B_0)^{-1} C_s A_0 x_s} \rightarrow \dot{S}_s = 0$

$\rightarrow$ remains on the sliding surface $S_s(x_s) = 0$

- Analyze the motion outside the sliding surface, $S_s(x_s) \neq 0$

Define $V(S_s(x_s)) = \frac{1}{2} S_s(x_s)^T S_s(x_s)$

Require $\dot{V}(S_s(x_s)) = S_s(x_s)^T \dot{S}_s(x_s) = S_s^T(x_s)\left(C_s A_0 x_s + C_s B_0 u_s\right) < 0$,

which is guaranteed by $\underline{u_s(t) = u_s^{eq}(t) - (C_s B_0)^{-1} \underbrace{\mathrm{diag}(\eta)}_{r\times r,\ \text{positive diag elts}} \overbrace{\mathrm{Sgn}(S_s(x_s))}^{= (\mathrm{sgn}(S_1(x_s)), \cdots)}}$

② Continuous closed-loop full system

(outside sliding surface) $\begin{cases} \dot{x}_s = A_{11} x_s + A_{12} z + B u_s, & t \neq \tau_k \\ \varepsilon \dot{z} = A_{21} x_s + A_{22} z, & t \neq \tau_k \\ x_s(0) = x_{s0}, \quad z(0) = z_0 \end{cases}$

(on the sliding surface) $S_s(x_s)=0 \Rightarrow u_s(t)=u_s^{eq}(t)$ $\quad (\mathrm{sgn}(S_s(x_s))=0)$

$$\begin{cases} \dot{x}_s = A_{11}x_s + A_{12}z + Bu_s^{eq}, & t \neq t_k \\ \varepsilon\dot{z} = A_{21}x_s + A_{22}z, & t \neq t_k \\ x_s(0)=x_{s0}, & z(0)=z_0 \end{cases}$$

Thm 11.1 Assume

① The reduced subsystem is stabilizable, A_{22} : Hurwitz

② For $t \neq t_k$, $\exists\ a_{21}, a_{22} > 0$:

$$-2(z-h(x))^T P \dot{h}(x) \le a_{21}x^Tx + a_{22}(z-h(x))^T(z-h(x)),$$

where P : $m\times m$ positive-definite, $A_{22}^TP + PA_{22} = -I$

③ $\exists\ \varepsilon^* > 0$: $-\tilde{A}$: M-matrix (pg 52. Def 2.36)

where $\tilde{A} = \begin{bmatrix} \frac{1}{\alpha_2}\max\{\mathrm{Re}[\lambda(\tilde{A}_{11})]\} & \frac{1}{2\gamma_1\beta_1} \\ \frac{a_{21}}{\alpha_1} & -\left(\frac{1}{\beta_2\varepsilon^*} - \frac{a_{22}}{\beta_1}\right) \end{bmatrix}$,

where $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1 > 0$ defined later, $\tilde{A}_{11} = A_{11} + \frac{\gamma_1}{2}A_{12}A_{12}^T - A_{12}A_{22}^{-1}A_{21} - B_1(CB_0)^{-1}CA_0$

④ For $i=1,\cdots,k$, $t_i - t_{i-1} > \frac{1}{\nu}\log(\alpha_{1i} + \alpha_{2i} + \beta_{2i})$ where

$\nu, \alpha_{1i}, \alpha_{2i}, \beta_{2i} > 0$ and $\alpha_{1i} + \alpha_{2i} + \beta_{2i} > 1$.

$\Rightarrow$ For each $\varepsilon \in (0, \varepsilon^*]$, the full impulsive system becomes globally exp. stable.

pf) $x(t) = x(t; t_0, x_0, z_0)$, $z(t) = z(t; t_0, x_0, z_0)$: solution

$V(x) = \frac{1}{2}x^Tx$, $W(z-h(x)) = (z-h(x))^TP(z-h(x))$: Lyapunov ftn candidates for fast, slow subsystems.

For $\alpha_1 \le \frac{1}{2}$, $\alpha_2 \ge \frac{1}{2}$, $\beta_1 \le \lambda_{min}(P)$, $\beta_2 \ge \lambda_{max}(P)$,

$$\begin{cases} \alpha_1 \|x\|^2 \le V(x) \le \alpha_2 \|x\|^2, \\ \beta_1 \|z-h(x)\|^2 \le W(z-h(x)) \le \beta_2 \|z-h(x)\|^2 \end{cases}$$

time derivative of V along x during sliding motion ($\varsigma = 0$)

$$: \dot V(x) = x^T \dot x = x^T (A_{11} x + A_{12} z + B u_s^{eq})$$

$$= x^T (A_{11} x + A_{12}(z-h(x)) + A_{12} h(x) - B_1 (CB_0)^{-1} C A_0 x)$$

$$\le x^T (A_{11} + \frac{\gamma_1}{2} A_{12} A_{12}^T - A_{12} A_{22}^{-1} A_{21} - B_1 (CB_0)^{-1} C A_0) x + \frac{1}{2\gamma_1} (z-h(x))^T (z-h(x))$$

$$\le \frac{1}{\alpha_2} \max\{Re[\lambda(\tilde A_{11})]\} V(x) + \frac{1}{2\gamma \beta_1} W(z-h(x))$$

time derivative of W along z during sliding motion

$$: \dot W(z-h(x)) \le -\left(\frac{1}{\beta_2 \varepsilon} - \frac{a_{22}}{\beta_1}\right) W(z-h(x)) + \frac{a_{21}}{\alpha_1} V(x)$$

$$\Rightarrow \begin{pmatrix} \dot V(x) \\ \dot W(z-h(x)) \end{pmatrix} \le \tilde A \begin{pmatrix} V(x) \\ W(z-h(x)) \end{pmatrix}, \quad -\tilde A : \text{M-matrix by } 3$$

$$\Rightarrow \exists \xi > 0 : \forall t \in [\tau_k, \tau_{k+1}),$$

$$\left.\begin{cases} V(x(t)) \le (\|V(\tau_k)\| + \|W(\tau_k)\|) e^{-\xi(t-\tau_k)} \\ W((z-h(x))(t)) \le (\|V(\tau_k)\| + \|W(\tau_k)\|) e^{-\xi(t-\tau_k)} \end{cases}\right) \boxed{1}$$

$$(V(\tau_k) = V(x(\tau_k)),\ W(\tau_k) = W((z-h(x))(\tau_k)))$$

$$\text{At } t = \tau_k, \quad V(x(\tau_k)) \le \alpha_{1k} V(x(\tau_k^-)), \quad \alpha_{1k} = \lambda_{max}^2 (I + E_k) \quad) \boxed{2}$$

$$W((z-h(x))(\tau_k)) = \cdots \le \beta_{2k} W((z-h(x))(\tau_k^-)) + \alpha_{2k} V(x(\tau_k^-)) \quad) \boxed{3}$$

$t \in [t_0, \tau_1) \longrightarrow \begin{cases} V(x(t)) \le \left(\|v(t_0)\| + \|w(t_0)\|\right) e^{-\xi(t-t_0)} & \boxed{1} \\ W((z-h(x))(t)) \le \quad " \end{cases}$

$t \in [\tau_1, \tau_2) \longrightarrow \begin{cases} V(x(t)) \le \left(\|v(\tau_1)\| + \|w(\tau_1)\|\right) e^{-\xi(t-\tau_1)} & \boxed{1} \\ \quad \le (\alpha_1 + \beta_{21} + \alpha_{21})(\|v(t_0)\| + \|\ \ \|) e^{-\xi(\tau_1 - t_0)} e^{-\xi(t-\tau_1)} & \boxed{2}, \boxed{3} \\ \quad = (\alpha_{11} + \beta_{21} + \alpha_{21})(\|\ \ \| + \|\ \ \|) e^{-\xi(t-t_0)} \\ W((z-h(x))(t)) \le \quad " \end{cases}$

$\therefore \ \forall t \in [t_0, \tau_k)$, $\begin{cases} V(x(t)) \le \prod_{i=1}^{k-1} (\alpha_{1i} + \beta_{2i} + \alpha_{2i}) \left(\|v(t_0)\| + \|w(t_0)\|\right) e^{-\xi(t-t_0)} \\ W((z-h(x))(t)) \le \quad " \end{cases}$

Choose $0 < \nu < \xi$. By ④, $\begin{cases} V(x(t)) \le C\left(\|v(t_0)\| + \|w(t_0)\|\right) e^{-(\xi-\nu)(t-t_0)} \\ W((z-h(x))(t)) \le \quad " \end{cases}$, $\forall t \ge t_0$

$\Rightarrow \ \|x(t)\| + \|z(t)\| \le k\left(\|x(t_0)\| + \|z(t_0)\|\right) e^{-(\xi-\nu)(t-t_0)/2}$, $\forall t \ge t_0$

(similar method in the pf of thm 9.1)

$$
(S)\begin{cases}
\dot{x} = A_{11}x + A_{12}z + Bu, & t \neq \tau_k \\
\varepsilon\dot{z} = A_{21}x + A_{22}z, & t \neq \tau_k \\
x(t) = (I + E_k)\,x(t^-), & t = \tau_k \\
z(t) = (I + F_k)\,z(t^-), & t = \tau_k \\
x(0) = x_0, \quad z(0) = z_0
\end{cases}
$$

We can observe $y = Dx$, $D \in \mathbb{R}^{l \times n}$.

Design sliding mode observer through the reduced slow system.

Define the state estimate impulsive SPS :

$$
\begin{cases}
\dot{\hat{x}} = A_{11}\hat{x} + A_{12}\hat{z} + Bu + Lv(\hat{y} - y), & t \neq \tau_k \\
\varepsilon\dot{\hat{z}} = A_{21}\hat{x} + A_{22}\hat{z}, & t \neq \tau_k \\
\hat{x}(t) = (I + E_k)\,\hat{x}(t^-), & t = \tau_k \\
\hat{z}(t) = (I + F_k)\,\hat{z}(t^-), & t = \tau_k \\
\hat{x}(0) = \hat{x}_0, \quad \hat{z}(0) = \hat{z}_0
\end{cases}
$$

L : observer gain matrix, v : r-dim observer to be designed.
$\cap$
$\mathbb{R}^{n \times r}$

error states $e_x = \hat{x} - x$, $e_z = \hat{z} - z$ satisfy

$$
\begin{cases}
\dot{e}_x = A_{11}e_x + A_{12}e_z + L\bar{v}(e_x), & t \neq \tau_k \\
\varepsilon\dot{e}_z = A_{21}e_x + A_{22}e_z, & t \neq \tau_k \\
e_x(t) = (I + E_k)\,e_x(t^-), & t = \tau_k \\
e_z(t) = (I + F_k)\,e_z(t^-), & t = \tau_k
\end{cases}
$$

$e_x(0) = e_{x0}$, $e_z(0) = e_{z0}$

$\bar{v}(e_x) = v(De_x)$

Setting $\varepsilon = 0$,

$\boxed{\text{reduced error subsystem}}$ $\dot{e}_{x_s} = A_e e_{x_s} + L_s \bar{v}(e_{x_s})$,

where $e_z = h(e_x) = -A_{22}^{-1} A_{21} e_x$, $A_e = A_{11} - A_{12} A_{22}^{-1} A_{21}$, $L_s = L$

Sliding mode error surface : $S_e(e_{x_s}) = C_e e_{x_s}$, $C_e \in \mathbb{R}^{r \times n}$

$\bar{v}_{eq}(e_{x_s}) = -(C_e L_s)^{-1} C_e A_e e_{x_s}$

$\dot{e}_{x_s} = (I - L_s (C_e L_s)^{-1} C_e) A_e e_{x_s} =: A_e^{eq} e_{x_s}$

$V(S_e(e_x)) = \frac{1}{2} S_e(e_{x_s})^T S_e(e_{x_s})$

$\rightarrow \dot{V}(S_e) < 0$ if $\bar{v}(e_{x_s}) = \bar{v}_{eq}(e_{x_s}) - (C_e L_s)^{-1} \mathrm{diag}(\eta)\, \mathrm{Sgn}(S_e(e_{x_s}))$

sliding mode control law

Thm 11.2 Assume

① Reduced slow, fast subsystems are observable

② $\exists \varepsilon^* > 0$: $-\tilde{A}$: M-matrix, $\tilde{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & -(\frac{1}{\varepsilon^*} - a_{22}) \end{bmatrix}$

③ $t_i - t_{i-1} > \frac{1}{\nu} \log(\alpha_{1i} + \alpha_{2i} + \beta_{2i})$, $\nu, \alpha_{1i}, \alpha_{2i}, \beta_{2i} > 0$, $\alpha_{1i} + \alpha_{2i} + \beta_{2i} > 1$

$\Rightarrow$ closed-loop full-order error system : globally exponentially stable, $\forall \varepsilon \in (0, \varepsilon^*]$

pf) Similar to thm 11.1